AN INTRODUCTION TO STOCHASTIC MODELS FOR CLAIMS RESERVING

Wüthrich, M.V., Merz, M. (2008), Stochastic Claim Reserving Methods in Insurance, Wiley

Agenda

- Introduction and notation
- Prediction and prediction error
- The Chain-Ladder (CL) model
- The Bornhuetter-Ferguson model
- Poisson derivation of the CL algorithm

INTRODUCTION AND NOTATION

Claims reserve for unsettled claims of past exposure:

- IBNeR (Incurred But Not enough Reported) or IBNS (Incurred But Not Settled)
- IBNyR (Incurred But Not yet Reported) or IBNR (Incurred But Not Reported)

Most of the classical claims reserving methods estimate the claims reserve for both the reported claims and the IBNyR at the same time.

Run-off table

$i =$ origin year or accident year, year of occurrence

$j =$ development year

$i \in \{0,1,...,I\}$ $\quad j \in \{0,1,...,J\}$ $\qquad J \le I$

<u>Observations</u> available at the end of year I:

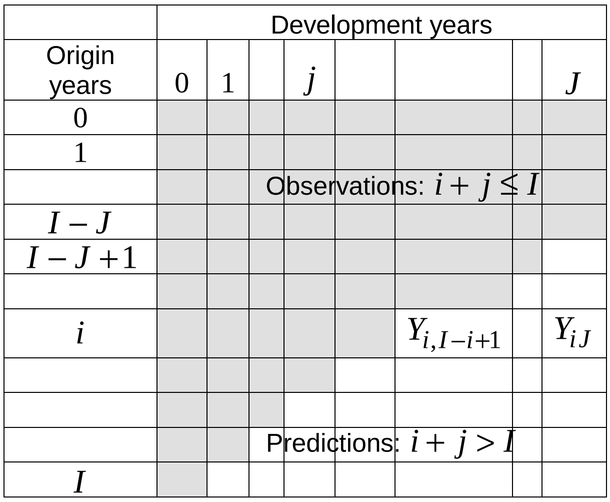
$$\{y_{ij} : \quad i + j \le I, \quad j = 0,\dots,J\}$$

	Development years							
Origin years	0	1		j				J
0								
1								
			Observations: $i + j \le I$					
$I - J$								
$I - J + 1$								
i	y_{i0}			y_{ij}	$y_{i,I-i}$			
I								

Y_{ij} **incremental claims** in development year j, for claims with origin year i; referred to the accounting year $i+j$

$C_{ij} = \sum_{k=0}^{j} Y_{ik}$ **cumulative claims** for origin year i after j development years

Incremental claims Y_{ij}	**Cumulative claims** C_{ij}
payments in cell (i, j)	cumulative payments
payments for finalized claims	cumulative payments for finalized claims
payments for unit of exposure (e.g. number payments in cell (i, j), number of policies, or number of claims, or earned premiums in origin year i)	cumulative payments for unit of total exposure
number of reported claims with delay j	claims incurred (cumulative payments + case estimate for unfinalized claims)
number of claims payments in cell (i, j)	total number of reported claims
	total number of payments



The lower triangle needs to be **estimated** or **predicted**

$$\{Y_{ij}: \quad i+j > I, \quad i \le I, \quad j \le J\}$$

In the following, let

Y_{ij} incremental payments

Then

C_{iJ} **ultimate claim amount** of origin year i

$R_i = \sum_{j=I-i+1}^{J} Y_{ij} = C_{iJ} - C_{i,I-i}$ **outstanding loss liabilities** for origin year i

Remark: R_i need to be predicted.

In WM the predictors for the R_i are named **claims reserves**; for brevity, we name claims reserves the R_i and call estimators of the claims reserves the corresponding predictors.

Remarks:

1) loss reserving models are applied for estimating/predicting the outstanding loss liabilities
2) different methods applied to the same aggregated data (payments or claims incurred, number of claims and claims averages statistics, indexed or unindexed claims data, ...) lead to different results
3) the claims figures in the claims development triangles (paid or incurred data) include the allocated loss adjustment expense (ALAE); the unallocated loss adjustment expenses (ULAE) need to be estimated separately
4) in classical claims reserving literature a point estimate of the outstanding loss liabilities is provided by applying an algorithm
5) stochastic claims reserving model are introduced:

    to justify claims reserving algorithms;

    to quantify the uncertainty inherent to the best estimate of the outstanding loss liabilities
6) we assume $I = J$ and denote $t = I = J$ the maximum development year
    - available information at the end of year t: $\{y_{ij} : \quad i + j \le t, \quad j = 0,\ldots,t\}$
    - we have to predict the outstanding loss liabilities for origin years $i = 1,...,t$

PREDICTION AND PREDICTION ERROR

Let

t the maximum development year

Y_{ij} incremental payments $i, j = 0,1,...,t$

Probabilistic assumptions on Y_{ij}, $i, j = 0,1,...,t$:

- parametric models: the distributions of the r.v. Y_{ij} belong to a specific family of distributions
- semi-parametric models: the distributions of the r.v. Y_{ij} are not completely specified; only some moment structures a given.

	Development years							
Origin years	0	1		j				t
0								
1	Upper triangle of observations							
i	Y_{i0}			Y_{ij}		$Y_{i,t-i+1}$		Y_{it}
			Lower triangle of predictions					
t								

➢ **Estimation problem**: given the data $\{y_{ij} : \quad i + j \le t, \quad j = 0,\ldots,t\}$ in the upper triangle, the model parameters need to be estimated.

➢ **Prediction problem**: the random variables in the lower triangle, or some functions of them, need to be predicted.

Let the r.v. W be a function of the r.v. in the lower triangle

$$W = \bar{f}\left(Y_{ij} : i + j > t\right)$$

e.g. $W = \sum_{i=1}^{t} \sum_{j=t-i+1}^{t} Y_{ij}$ the total outstanding loss liabilities or claims reserve.

When a probabilistic model for the r.v. Y_{ij}, $i, j = 0,1,...,t$ has been estimated, it can be used to predict the r.v. W throughout an estimator

$$\tilde{W} = f\left(Y_{ij} : i + j \leq t\right)$$

that is a function of the r.v. in the upper triangle. The estimator $\tilde{W}$ is also called **predictor** of W.

The estimate

$$\hat{W} = f\left(y_{ij} : i + j \leq t\right)$$

gives the prediction of W.

The estimator should satisfy some properties, such as unbiasedness (i.e. $E(\tilde{W}) = E(W)$), …

To quantify the quality of the estimator $\tilde{W}$ we consider the **prediction error of the estimator** given by the root of the Mean Square Error of Prediction.

The **Mean Square Error of Prediction (MSEP)** can be evaluated unconditional or conditional on the set of r.v. $\mathscr{D}_t = \{Y_{ij} : i + j \leq t\}$ for which the observations are available.

Unconditional mean square error of prediction

$$MSEP(\tilde{W}) = E\left[(\tilde{W} - W)^2\right]$$

The **prediction error of the estimator** (root of the Mean Square Error of Prediction) is denoted by RMSEP.

Remark:

$E\left[(\tilde{W} - E(W))^2\right]$ does not tell us anything about the quality of the estimate provided by $\tilde{W}$ for a specific realization of $\mathscr{D}_t$.

If $\tilde{W}$ and W are independent

$$MSEP(\tilde{W}) = E\left[(\tilde{W} - W)^2\right] = var(W) + E\left[(\tilde{W} - E(W))^2\right]$$

where

- $var(W)$ is the **process variance** which describes the intrinsic "variability" of W; its square root quantify the so-called <u>process risk</u>
- $E\left[(\tilde{W} - E(W))^2\right]$ is the **average parameter estimation error** which reflects the uncertainty in the estimation of the parameters of the model; its square root quantify the so-called <u>estimation risk</u>.

If $\tilde{W}$ is an unbiased estimator for $E(W)$ (i.e. $E(W) = E(\tilde{W})$)

$$MSEP(\tilde{W}) = var(W) + E\left[(\tilde{W} - E(\tilde{W}))^2\right] = var(W) + var(\tilde{W})$$

To quantify the quality of the estimator $\tilde{W}$ for a specific realization of $\mathscr{D}_t$ we consider the

Conditional mean square error of prediction

$$MSEP_{\mathscr{D}_t}(\tilde{W}) = E\left[(\tilde{W} - W)^2 \middle| \mathscr{D}_t\right] = var(W|\mathscr{D}_t) + \left(\tilde{W} - E(W|\mathscr{D}_t)\right)^2$$

where

- $var(W|\mathscr{D}_t)$ is the conditional process variance which describes the intrinsic "variability" of W; its square root quantify the so-called process risk
- $\left(\tilde{W} - E(W|\mathscr{D}_t)\right)^2$ is the **parameter estimation error** which reflects the uncertainty in the estimation of the parameters of the model; its square root quantify the so-called <u>estimation risk</u>.

Remarks:

- $MSEP_{\mathscr{D}_t}(\tilde{W}) = E\left[(\tilde{W} - W)^2 \middle| \mathscr{D}_t\right]$ is a r.v.

- The unconditional MSEP is the expectation of the conditional MSEP on $\mathscr{D}_t$, in fact

$$MSEP(\tilde{W}) = E\left[(\tilde{W} - W)^2\right] = E\left\{E\left[(\tilde{W} - W)^2 \middle| \mathscr{D}_t\right]\right\} = E\left\{MSEP_{\mathscr{D}_t}(\tilde{W})\right\}$$

- $\tilde{W} = f\left(Y_{ij} : i + j \le t\right)$ is a $\mathscr{D}_t$-measurable estimator for $E(W|\mathscr{D}_t)$ and a predictor for W.

- We have to estimate the parameter estimation error,

$$\left(\tilde{W} - E(W|\mathscr{D}_t)\right)^2$$

  because $E(W|\mathscr{D}_t)$ is unknown and we use $\tilde{W}$ as an estimate.

THE CHAIN-LADDER MODEL

In actuarial literature the Chain-Ladder (CL) method is often understood as a purely computational algorithm and it leaves the question open as to which probabilistic models lead to that algorithm.

Let

Y_{ij} incremental payments, $i, j = 0,\ldots,t$

$C_{ij} = \sum_{k=0}^{j} Y_{ik}$ cumulative claims for origin year i after j development years

Distribution-free CL model (Mack (1993); Wüthrich, Merz (2008))

CL1) the random vectors $(C_{i0},\ldots,C_{it})$, $i = 0,\ldots,t$ are stochastically independent

CL2) $(C_{i0},...,C_{it})$ form a Markov chain, $i = 0,\ldots,t$
There exist development factors $f_j > 0$, $j = 0,\ldots,t-1$,
and variance parameters $\sigma_j^2 > 0$, $j = 0,\ldots,t-1$,
such that for all $i = 0,\ldots,t$ and for all $j = 0,\ldots,t$, we have

$$E\left(C_{ij} \middle| C_{i0},\ldots,C_{i,j-1}\right) = E\left(C_{ij} \middle| C_{i,j-1}\right) = f_{j-1} C_{i,j-1}$$
$$var\left(C_{ij} \middle| C_{i0},\ldots,C_{i,j-1}\right) = var\left(C_{ij} \middle| C_{i,j-1}\right) = \sigma_{j-1}^2 C_{i,j-1}$$

Remarks:

- We make assumptions only on the first two moments and not on the explicit distribution of C_{ij} given $C_{i,\,j-1}$.
- The factors f_j are called link ratios, development factors, CL factors of age-to-age factors. They describe how we link successive cumulative claims.

Given $\mathscr{D}_t = \{Y_{ij} : i + j \le t\}$, under the assumptions CL1) and CL2), recursively we have

$$\Rightarrow \qquad E(C_{it} | \mathscr{D}_t) = C_{i,t-i} \cdot f_{t-i} \cdot \ldots \cdot f_{t-1} \qquad i = 0, \ldots, t$$

Hence, we get that the outstanding claims liabilities of origin year i at time t, based on $\mathscr{D}_t$, are predicted by

$$E(C_{it} | \mathscr{D}_t) - C_{i,t-i} = C_{i,t-i} \cdot (f_{t-i} \cdot \ldots \cdot f_{t-1} - 1) \qquad i = 0, \ldots, t$$

The CL factors f_j are unknown and need to be estimated.

It can be shown that the following estimators are unbiased and uncorrelated estimators of the parameters f_j

	0		j	$j+1$			t
0							
1							
$t-j-1$							
t							

$$\tilde{f}_j = \frac{\sum_{i=0}^{t-j-1} C_{i,\,j+1}}{\sum_{i=0}^{t-j-1} C_{ij}} = \sum_{i=0}^{t-j-1} \frac{C_{ij}}{\sum_{k=0}^{t-j-1} C_{kj}} \frac{C_{i,\,j+1}}{C_{ij}} \qquad j = 0, \ldots, t-1$$

The **CL estimator for the ultimate claims** C_{it} is

$$\tilde{C}_{it}^{CL} = C_{i,t-i} \prod_{j=t-i}^{t-1} \tilde{f}_j$$

and the **CL estimator for the claims reserve** $R_i = C_{it} - C_{i,t-i}$ is

$$\tilde{R}_i^{CL} = \tilde{C}_{it}^{CL} - C_{i,t-i}$$

Remarks:

- $\tilde{C}_{it}^{CL}$ in an unbiased estimator of $E(C_{it})$

It can be shown that the following estimators are unbiased estimators of the parameters $\sigma_j^2,\ j = 0,\ldots,t-2$.

$$\tilde{\sigma}_j^2 = \frac{1}{t-j-1} \sum_{i=0}^{t-j-1} C_{ij} \left(\frac{C_{i,j+1}}{C_{ij}} - \tilde{f}_j \right)^2 \qquad j = 0,\ldots,t-2$$

It can be interpreted as a weighted average of the square of the residuals.

As for the parameter σ_{t-1}^2, if we do not have enough data (i.e. we do not have $I > J$) we cannot estimate it similarly.

Since the estimates $\hat{\sigma}_0^2,\ldots,\hat{\sigma}_{t-2}^2$ are generally decreasing, an estimate for σ_{t-1}^2 can be obtained by extrapolation according to some formulas such as (see WM):

$$\hat{\sigma}_{t-1}^2 = min\left(\frac{\hat{\sigma}_{t-2}^4}{\hat{\sigma}_{t-3}^2}, \hat{\sigma}_{t-3}^2, \hat{\sigma}_{t-2}^2 \right)$$

CL prediction and prediction error for a single origin year

Denote by $\hat{f}_j$ the estimates provided by the estimators $\tilde{f}_j$ of the CL factors f_j

$$\hat{f}_j = \frac{\sum_{i=0}^{t-j-1} c_{i,j+1}}{\sum_{i=0}^{t-j-1} c_{ij}} \qquad j = 0,\ldots,t-1$$

The **CL estimate for the ultimate claims** C_{it} is

$$\hat{C}_{it}^{CL} = c_{i,t-i} \prod_{j=t-i}^{t-1} \hat{f}_j$$

and the **CL estimates for the claims reserve** $R_i = C_{it} - C_{i,t-i}$ is

$$\hat{R}_i^{CL} = \hat{C}_{it}^{CL} - c_{i,t-i}$$

Remark:

➢ Since the estimates $\hat{f}_j$ are those of the CL method, the estimates $\hat{R}_i$ are the CL estimates of the claims reserves

We have to evaluate the **conditional mean square error of prediction** of the CL estimators of the claims reserves $\tilde{R}_i^{CL} = \tilde{C}_{it}^{CL} - C_{i,t-i}$

$$MSEP_{\mathscr{D}_t}(\tilde{R}_i^{CL}) = E\left[(\tilde{R}_i^{CL} - R_i)^2 \middle| \mathscr{D}_t\right] = E\left[(\tilde{C}_{it}^{CL} - C_{i,t-i} - C_{it} + C_{i,t-i})^2 \middle| \mathscr{D}_t\right]$$

$$= E\left[(\tilde{C}_{it}^{CL} - C_{it})^2 \middle| \mathscr{D}_t\right] = MSEP_{\mathscr{D}_t}(\tilde{C}_{it}^{CL}) = var(C_{it}|\mathscr{D}_t) + \left(\tilde{C}_{it}^{CL} - E(C_{it}|\mathscr{D}_t)\right)^2$$

An estimate of the <u>conditional process variance</u> $var(C_{it}|\mathscr{D}_t)$ is given by

$$v\hat{a}r(C_{it}|\mathscr{D}_t) = (\hat{C}_{it}^{CL})^2 \sum_{j=t-i}^{t-1} \frac{\hat{\sigma}_j^2 / \hat{f}_j^2}{\hat{C}_{ij}^{CL}}$$

where

$$\hat{C}_{ij}^{CL} = c_{i,t-i} \prod_{h=t-i}^{j-1} \hat{f}_h$$

is the CL estimate of $E\left[C_{ij} \middle| C_{i,t-i}\right]$

Much more difficult is to evaluate the <u>conditional parameter estimation error</u>, which provides an estimate for the accuracy of the CL factor estimators $\tilde{f}_j$.
In fact, we have

$$\left(\tilde{C}_{it}^{CL} - E(C_{it}|\mathscr{D}_t)\right)^2 = \left(C_{i,t-i} \prod_{j=t-i}^{t-1} \tilde{f}_j - C_{i,t-i} \prod_{j=t-i}^{t-1} f_j\right)^2 =$$

$$= C_{i,t-i}^2 \left(\tilde{f}_{t-i} \cdot \ldots \cdot \tilde{f}_{t-1} - f_{t-i} \cdot \ldots \cdot f_{t-1}\right)^2$$

The quantity in brackets cannot be calculated directly because, whereas at time t the estimates $\hat{f}_j$ are known, the parameters f_j are not and we actually use $\hat{f}_j$ to estimate them.

Hence, the conditional parameter estimation error needs to be estimated in some way.

The idea is to analyze the variability of the CL factor estimators $\hat{f}_j$ around the values f_j and then to derive an analytical formula that provide an estimate of the conditional parameter estimation error.

There are various approaches. By following the Mack (1993) approach, we can rewrite the quantity in brackets, on the right-hand side of the equation

$$\left(\tilde{C}_{it}^{CL} - E(C_{it}|\mathscr{D}_t)\right)^2 = C_{i,t-i}^2\left(\tilde{f}_{t-i}\cdot\ldots\cdot\tilde{f}_{t-1} - f_{t-i}\cdot\ldots\cdot f_{t-1}\right)^2$$

as follows

$$\left(\tilde{f}_{t-i}\cdot\ldots\cdot\tilde{f}_{t-1} - f_{t-i}\cdot\ldots\cdot f_{t-1}\right)^2 = \left(\sum_{j=t-i}^{t-1} T_j\right)^2 = \sum_{j=t-i}^{t-1} T_j^2 + 2\sum_{t-i\le j<k\le t-1}^{t-1} T_j T_k$$

where

$$T_j = \tilde{f}_{t-i}\cdot\tilde{f}_{t-i+1}\cdot\ldots\cdot\tilde{f}_{j-1}\left(f_j - \tilde{f}_j\right)f_{j+1}\cdot\ldots\cdot f_{t-1}\,, \qquad j = t-i,\ldots,t-1.$$

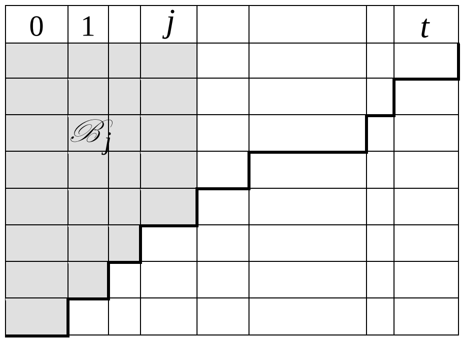
The idea is to estimate

$$\left(\tilde{f}_{t-i}\cdot\ldots\cdot\tilde{f}_{t-1} - f_{t-i}\cdot\ldots\cdot f_{t-1}\right)^2$$

through the evaluation of the expectation of the r.v.

$$\sum_{j=t-i}^{t-1} T_j^2 + 2\sum_{t-i\le j<k\le t-1}^{t-1} T_j T_k$$

Denoted by $\mathscr{B}_j = \{Y_{ik} : i + k \le t,\ 0 \le k \le j\}$



$$\Rightarrow\ E(T_j|\mathscr{B}_j) = \hat{f}_{t-i}\cdot\ldots\cdot\hat{f}_{j-1}E(f_j - \tilde{f}_j|\mathscr{B}_j)\,f_{j+1}\cdot\ldots\cdot f_{t-1} = 0$$

$\Rightarrow\ E(T_k T_j|\mathscr{B}_j) = 0$ for $k < j$

$\Rightarrow\ E(T_j^2|\mathscr{B}_j) = \hat{f}_{t-i}^2\cdot\hat{f}_{t-i+1}^2\cdot\ldots\cdot\hat{f}_{j-1}^2 var\left(\tilde{f}_j|\mathscr{B}_j\right)f_{j+1}^2\cdot\ldots\cdot f_{t-1}^2$ and $var\left(\tilde{f}_j|\mathscr{B}_j\right) = \dfrac{\sigma_j^2}{\sum_{h=0}^{t-j-1} C_{hj}}$

Hence, an estimate for the conditional parameter estimation error

$$\left(\tilde{C}_{it}^{CL} - E(C_{it}|\mathscr{D}_t)\right)^2 = C_{i,t-i}^2\left(\tilde{f}_{t-i}\cdot\ldots\cdot\tilde{f}_{t-1} - f_{t-i}\cdot\ldots\cdot f_{t-1}\right)^2$$

is given by

$$c_{i,t-i}^2 \sum_{j=t-i}^{t-1} \hat{f}_{t-i}^2\cdot\hat{f}_{t-i+1}^2\cdot\ldots\cdot\hat{f}_{j-1}^2 \frac{\hat{\sigma}_j^2\Big/\hat{f}_j^2}{\sum_{h=0}^{t-j-1} c_{hj}} \hat{f}_j^2\cdot\hat{f}_{j+1}^2\cdot\ldots\cdot\hat{f}_{t-1}^2 = c_{i,t-i}^2\left(\prod_{j=t-i}^{t-1}\hat{f}_j^2\right)\sum_{j=t-i}^{t-1}\frac{\hat{\sigma}_j^2\Big/\hat{f}_j^2}{\sum_{h=0}^{t-j-1} c_{hj}}$$

$$= (\hat{C}_{it}^{CL})^2 \sum_{j=t-i}^{t-1}\frac{\hat{\sigma}_j^2\Big/\hat{f}_j^2}{\sum_{h=0}^{t-j-1} c_{hj}}$$

Therefore, we get the following estimate for the conditional mean square error of prediction

$$M\hat{S}EP_{\mathscr{D}_t}(\tilde{R}_i^{CL}) = (\hat{C}_{it}^{CL})^2\sum_{j=t-i}^{t-1}\frac{\hat{\sigma}_j^2 / \hat{f}_j^2}{\hat{C}_{ij}^{CL}} + (\hat{C}_{it}^{CL})^2\sum_{j=t-i}^{t-1}\frac{\hat{\sigma}_j^2 / \hat{f}_j^2}{\sum_{h=0}^{t-j-1} c_{hj}} =$$

$$= (\hat{C}_{it}^{CL})^2\sum_{j=t-i}^{t-1}\frac{\hat{\sigma}_j^2}{\hat{f}_j^2}\left(\frac{1}{\hat{C}_{ij}^{CL}} + \frac{1}{\sum_{h=0}^{t-j-1} c_{hj}}\right)$$

CL prediction and prediction error for aggregated origin years

Denoted by R the **total claims reserve**

$$R = \sum_{i=1}^{t} R_i$$

the **CL estimator for the total claims reserve** is

$$\tilde{R}^{CL} = \sum_{i=1}^{t}\tilde{R}_i^{CL} = \sum_{i=1}^{t}\left(\tilde{C}_{it}^{CL} - C_{i,t-i}\right)$$

We have to evaluate the **conditional mean square error of prediction** of the CL estimator of the total claims reserve $\tilde{R}^{CL}$

$$MSEP_{\mathscr{D}_t}(\tilde{R}^{CL}) = E\left[\left(\sum_{i=1}^{t}\tilde{R}_i^{CL} - \sum_{i=1}^{t}R_i\right)^2\middle|\mathscr{D}_t\right] = E\left[\left(\sum_{i=1}^{t}\tilde{C}_{it}^{CL} - \sum_{i=1}^{t}C_{it}\right)^2\middle|\mathscr{D}_t\right] = MSEP_{\mathscr{D}_t}\left(\sum_{i=1}^{t}\tilde{C}_{it}^{CL}\right)$$

Consider two different origin years $i < k$

$$MSEP_{\mathscr{D}_t}\left(\tilde{C}_{it}^{CL} + \tilde{C}_{kt}^{CL}\right) = E\left[(\tilde{C}_{it}^{CL} + \tilde{C}_{kt}^{CL} - C_{it} - C_{kt})^2 \middle| \mathscr{D}_t\right]$$

$$= var(C_{it} + C_{kt} | \mathscr{D}_t) + \left(\tilde{C}_{it}^{CL} + \tilde{C}_{kt}^{CL} - E(C_{it} + C_{kt} | \mathscr{D}_t)\right)^2$$

Because of the independence assumption (CL1)

$$var(C_{it} + C_{kt} | \mathscr{D}_t) = var(C_{it} | \mathscr{D}_t) + var(C_{kt} | \mathscr{D}_t)$$

For the conditional parameter estimation error we have

$$\left(\tilde{C}_{it}^{CL} + \tilde{C}_{kt}^{CL} - E(C_{it} | \mathscr{D}_t) - E(C_{kt} | \mathscr{D}_t)\right)^2 = \left(\tilde{C}_{it}^{CL} - E(C_{it} | \mathscr{D}_t)\right)^2 + \left(\tilde{C}_{kt}^{CL} - E(C_{kt} | \mathscr{D}_t)\right)^2 + $$
$$+ 2\left(\tilde{C}_{it}^{CL} - E(C_{it} | \mathscr{D}_t)\right)\left(\tilde{C}_{kt}^{CL} - E(C_{kt} | \mathscr{D}_t)\right)$$

Hence

$$MSEP_{\mathscr{D}_t}\left(\tilde{C}_{it}^{CL} + \tilde{C}_{kt}^{CL}\right) = MSEP_{\mathscr{D}_t}\left(\tilde{C}_{it}^{CL}\right) + MSEP_{\mathscr{D}_t}\left(\tilde{C}_{kt}^{CL}\right) +$$
$$+ 2\left(\tilde{C}_{it}^{CL} - E(C_{it} | \mathscr{D}_t)\right)\left(\tilde{C}_{kt}^{CL} - E(C_{kt} | \mathscr{D}_t)\right)$$

So, in the $MSEP_{\mathscr{D}_t}(\tilde{R}^{CL})$ we have to estimate all the cross-products.

In WM the following estimate is suggested (Estimator 3.16):

$$M\hat{S}EP_{\mathscr{D}_t}(\tilde{R}^{CL}) = \sum_{i=1}^{t} M\hat{S}EP_{\mathscr{D}_t}(\tilde{R}_i^{CL}) + 2 \sum_{1 \le i < k \le t} \hat{C}_{it}^{CL} \hat{C}_{kt}^{CL} \sum_{j=t-i}^{t-1} \frac{\hat{\sigma}_j^2 / \hat{f}_j^2}{\sum_{h=0}^{t-j-1} c_{hj}}$$

$$= \sum_{i=1}^{t} \left\{ M\hat{S}EP_{\mathscr{D}_t}(\tilde{R}_i^{CL}) + \hat{C}_{it}^{CL} \left(\sum_{k=i+1}^{t} \hat{C}_{kt}^{CL} \right) \sum_{j=t-i}^{t-1} 2 \frac{\hat{\sigma}_j^2 / \hat{f}_j^2}{\sum_{h=0}^{t-j-1} c_{hj}} \right\}$$

THE BORNHUETTER-FERGUSON MODEL

Such as the CL method, also the Bornhuetter-Ferguson (BF) method is usually understood as a purely mechanical algorithm; however it is possible to define an appropriate underlying stochastic model which motivates the BF method.

Let

Y_{ij} incremental payments, $i, j = 0,\ldots,t$

$C_{ij} = \sum_{k=0}^{j} Y_{ik}$ cumulative claims for origin year i after j development years

BF model (Wüthrich, Merz (2008))

BF1) the random vectors $(C_{i0},\ldots,C_{it})$, $i = 0,\ldots,t$ are stochastically independent

BF2) There exist parameters $u_i > 0$, $i = 0,\ldots,t$, $b_j > 0$, $j = 0,\ldots,t$ with $b_t = 1$, such that for all $i = 0,\ldots,t$, $j = 0,\ldots,t-1$ and $k = 0,\ldots,t-j$, we have

$$E(C_{i0}) = u_i b_0$$

$$E\left(C_{i,\ j+k} \middle| C_{i0},\ldots,C_{ij}\right) = C_{ij} + u_i\left(b_{j+k} - b_j\right)$$

Under the assumptions BF1) and BF2), we have

$\Rightarrow$ $E(C_{ij}) = u_i b_j$ and $E(C_{it}) = u_i$ $i = 0,\ldots,t$

Remarks:

- The parameter u_i can be interpreted as the expected value of the ultimate claims for origin year i
- The parameters b_j, $j = 0,\ldots,t$, reflect the expected cumulative development pattern: b_j is the rate of the ultimate claims which is expected to be paid within development year j

Given $\mathscr{D}_t = \{Y_{ij} : i + j \le t\}$, under the assumptions BF1) and BF2), we have

$$\Rightarrow \qquad E\left(C_{it} \middle| \mathscr{D}_t\right) = C_{i,\ t-i} + u_i\left(1 - b_{t-i}\right) \qquad i = 0,\ldots,t$$

Hence, the outstanding claims liabilities of origin year i at time t based on $\mathscr{D}_t$ are predicted by

$$E\left(C_{it} \middle| \mathscr{D}_t\right) - C_{i,t-i} = u_i\left(1 - b_{t-i}\right) \qquad i = 0,\ldots,t$$

Denote by $\tilde{u}_i$, $i = 0,\dots,t$, and $\tilde{b}_j$, $j = 0,\dots,t$, some appropriate estimators of the parameters, then the **BF estimator for the ultimate claims** C_{it} is

$$\tilde{C}_{it}^{BF} = C_{i,\,t-i} + \tilde{u}_i \left(1 - \tilde{b}_{t-i}\right)$$

and the **BF estimator for the claims reserve** $R_i = C_{it} - C_{i,t-i}$ is

$$\tilde{R}_i^{BF} = \tilde{C}_{it}^{BF} - C_{i,t-i} = \tilde{u}_i \left(1 - \tilde{b}_{t-i}\right)$$

Remarks:

- Under the BF model, no dispersion hypotheses are assumed; for the evaluation of the MSEP we will see an approach developed within the GLM framework.
- Since $E(C_{it}) = u_i$, the estimates of the parameters u_i, $i = 0,\dots,t$, are called “initial” estimates of the ultimate claims; typically, these estimates are based on external data, e.g. from pricing or market information.
- As for the estimate of the development pattern, the CL development factors $\hat{f}_j$ are used.

In fact, from assumption CL2) we get

$$E(C_{ij}) = E\left[E\left(C_{ij} \middle| C_{i,\,j-1}\right)\right] = f_{j-1} E\left(C_{i,\,j-1}\right).$$

By iterating we have

$$E(C_{ij}) = E(C_{it}) \left(\prod_{k=j}^{t-1} f_k^{-1} \right),$$

where $\prod_{k=j}^{t-1} f_k^{-1}$ represents the rate of the expected ultimate claims paid within development year j.

Since from assumption BF2) we have $E(C_{ij}) = u_i b_j = E(C_{it}) b_j$, it seems plausible to estimate the parameter b_j by

$$\hat{b}_j^{CL} = \prod_{k=j}^{t-1} \hat{f}_k^{-1}$$

with $j = 0,\dots,t-1$ and $\hat{f}_k$, $k = 0,\dots,t-1$, the CL estimates of the CL factors f_j.

Comparison between BF and CL estimates

Let the **BF estimate for the claims reserve** $R_i = C_{it} - C_{i,t-i}$

$$\hat{R}_i^{BF} = \hat{u}_i \left(1 - \hat{b}_{t-i}^{CL}\right)$$

where

$\hat{u}_i$, $i = 0,\ldots,t$, are the "initial" estimates of the ultimate claims;

$\hat{b}_j^{CL} = \prod_{k=j}^{t-1} \hat{f}_k^{-1}$ with $\hat{f}_k$, $k = 0,\ldots,t-1$, the CL estimates of the CL factors f_j.

Since the CL claims reserve can be written as follows

$$\hat{R}_i^{CL} = \hat{C}_{it}^{CL} - c_{i,\,t-i} = \hat{C}_{it}^{CL} - \hat{C}_{it}^{CL}\,\hat{b}_{t-i}^{CL} = \hat{C}_{it}^{CL}\left(1 - \hat{b}_{t-i}^{CL}\right)$$

in the BF method we completely believe in our initial estimate $\hat{u}_i$; on the other side, in the CL method, the initial estimate is replaced by the estimate $\hat{C}_{it}^{CL}$,which is only based on the run-off observations.

We will see that such two "extreme positions" in the claims reserving problem can be combined ("credible claims reserves").

POISSON DERIVATION OF THE CL ALGORITHM

The Poisson model is mainly used for claims counts. However, since the maximum likelihood estimators of the parameters of the cross-classified Poisson model lead to the same reserve as the CL algorithm, the Poisson model is an alternative stochastic model (besides the distribution-free CL model) that can be used to motivate the CL reserves.

Let

Y_{ij} incremental payments, $i, j = 0,\ldots,t$

$C_{ij} = \sum_{k=0}^{j} Y_{ik}$ cumulative claims for origin year i after j development years

Poisson model

There exist parameters $a_i > 0$, $i = 0,\ldots,t$, $\gamma_j > 0$, $j = 0,\ldots,t$ such that the incremental payments Y_{ij} are independent, Poisson distributed with

$$E(Y_{ij}) = a_i \gamma_j \qquad \text{for all } i = 0,\ldots,t,\ j = 0,\ldots,t \qquad \text{and } \sum_{j=0}^{t} \gamma_j = 1$$

Under the Poisson model assumptions we have

$$\Rightarrow \qquad C_{it} = \sum_{k=0}^{t} Y_{ik} \qquad \text{is Poisson distributed with} \quad E(C_{it}) = a_i \qquad i = 0,\ldots,t$$

Remarks:

- a_i can be interpreted as the expected value of the ultimate claims for origin year i
- γ_j, $j = 0,\ldots,t$, define an expected “cash-flow pattern” over the different development periods j; in fact, γ_j can be interpreted as the rate of the ultimate claims, paid in the development period j.

  Moreover, such development pattern is independent of i, in fact

$$\frac{E(Y_{ij})}{E(Y_{i0})} = \frac{\gamma_j}{\gamma_0} \qquad j = 1,\ldots,t \qquad \text{is independent of } i.$$

Given $\mathscr{D}_t = \{Y_{ij} : i + j \le t\}$, under the Poisson model assumptions, we have

$$\Rightarrow \qquad E(C_{it} | \mathscr{D}_t) = C_{i,\ t-i} + a_i \sum_{j=t-i+1}^{t} \gamma_j = C_{i,\ t-i} + a_i \left(1 - \sum_{j=0}^{t-i} \gamma_j\right) \qquad i = 0,\ldots,t$$

We note the same expression implied by the BF assumptions:

$$E(C_{it} | \mathscr{D}_t) = C_{i,\ t-i} + u_i (1 - b_{t-i}) \qquad i = 0,\ldots,t$$

Maximum likelihood estimates of the parameters

Given the set of observations $\{y_{ij} : i + j \le t,\ j = 0,\ldots,t\}$ the likelihood function is

$$L(a_0,\ldots,a_t,\gamma_0,\ldots\gamma_t) = \prod_{i+j\le t} \left(\exp(-a_i\gamma_j) \frac{(a_i\gamma_j)^{y_{ij}}}{y_{ij}!} \right)$$

and the log-likelihood equations are

$$\begin{cases} \sum_{j=0}^{t-i} a_i \gamma_j = \sum_{j=0}^{t-i} y_{ij} & i = 0,\ldots,t \\ \sum_{i=0}^{t-j} a_i \gamma_j = \sum_{i=0}^{t-j} y_{ij} & j = 0,\ldots,t \end{cases}$$

under the constraint that $\sum_{j=0}^{t} \gamma_j = 1$.

We denote the solution of the log-likelihood equations as **maximum likelihood (ML) estimates** of the parameters of the Poisson model

$$\hat{a}_i^{POI}, \quad i = 0,\ldots,t, \qquad \hat{\gamma}_j^{POI}, \quad j = 0,\ldots,t$$

and denote

$$\tilde{a}_i^{POI}, \quad i = 0,\ldots,t, \qquad \tilde{\gamma}_j^{POI}, \quad j = 0,\ldots,t$$

the respective estimators.

The **Poisson ML estimators** for $E(Y_{ij}|\mathscr{D}_t) = E(Y_{ij}) == a_i \gamma_j$ and $E(C_{it}|\mathscr{D}_t)$ are

$$\tilde{Y}_{ij}^{POI} = \tilde{a}_i^{POI} \, \tilde{\gamma}_j^{POI}$$

and

$$\tilde{C}_{it}^{POI} = C_{i,\ t-i} + \tilde{a}_i^{POI} \left(1 - \sum_{j=0}^{t-i} \tilde{\gamma}_j^{POI} \right)$$

Remark:

It can be proved that the CL estimator

$$\tilde{C}_{it}^{CL} = C_{i,t-i} \prod_{j=t-i}^{t-1} \tilde{f}_j$$

and the Poisson ML likelihood estimator

$$\tilde{C}_{it}^{POI} = C_{i,\ t-i} + \tilde{a}_i^{POI} \left(1 - \sum_{j=0}^{t-i} \tilde{\gamma}_j^{POI} \right)$$

lead to the same reserve estimates.

Remarks:

- The distribution-free CL model and the Poisson model provide the same estimates of the claims development pattern

$$\hat{b}_j^{CL} = \prod_{k=j}^{t-1} \hat{f}_k^{\,-1} = \sum_{j=0}^{j} \tilde{\gamma}_k^{POI}$$

- If the CL claims development pattern is used to estimate the BF claims reserves

$$\hat{R}_i^{BF} = \hat{u}_i \left(1 - \hat{b}_{t-i}^{CL}\right)$$

  we note that, since the CL and the Poisson ML estimates of the development cash-flow pattern are the same, the reserve estimates only differ in the choice of the expected ultimate claims,

$$\hat{R}_i^{POI} = \hat{a}_i^{POI} \left(1 - \sum_{j=0}^{t-i} \hat{\gamma}_j^{POI} \right)$$

  in fact we have the initial estimate of the ultimate claims $\hat{u}_i$ for the BF reserve and the ML estimate $\hat{a}_i^{POI}$ for the Poisson model.

- Since $\hat{R}_i^{POI} = \hat{R}_i^{CL} = \hat{C}_{it}^{CL}\left(1 - \hat{b}_{t-i}^{CL}\right)$ then $\hat{a}_i^{POI} = \hat{C}_{it}^{CL}$.

Remark:

- The Poisson model implies that the increments Y_{ij} are non-negative.

  However in practical applications (e.g. in the case of claims incurred) we also observe negative increments, which indicates that the Poisson model is not appropriate.

  Anyway, the distribution-free CL model also applies for negative increments, as long as cumulative payments are positive.

- Under the Poisson model
  - the incremental payments Y_{ij} are independent Poisson distributed
  - $E(Y_{ij}) = a_i \gamma_j \Leftrightarrow \log(E(Y_{ij})) = \log(a_i) + \log(\gamma_j)$ for all $i = 0, \ldots, t$, $j = 0, \ldots, t$

  hence, we could think of a Generalized Linear Model.

This consideration leads us to the second part.