THE BORNHUETTER-FERGUSON MODEL
AND CREDIBLE CLAIMS RESERVES

Agenda

- Prediction error for the Bornhuetter-Ferguson model
- The Bornhuetter-Ferguson model and GLM
- A stochastic claims reserving model with random effects
- Prediction and prediction error for Credible claims reserves

PREDICTION ERROR FOR THE BORNHUETTER-FERGUSON MODEL

Let

Y_{ij} incremental payments, $i, j = 0,\ldots,t$

$C_{ij} = \sum_{k=0}^{j} Y_{ik}$ cumulative claims for origin year i after j development years

BF model (Wüthrich, Merz (2008))

BF1) the random vectors $(C_{i0},\ldots,C_{it})$, $i = 0,\ldots,t$ are stochastically independent

BF2) There exist parameters $u_i > 0$, $i = 0,\ldots,t$, $b_j > 0$, $j = 0,\ldots,t$ with $b_t = 1$, such that for all $i = 0,\ldots,t$, $j = 0,\ldots,t-1$ and $k = 0,\ldots,t-j$, we have

$$E(C_{i0}) = u_i b_0$$

$$E\left(C_{i,\ j+k} \middle| C_{i0},\ldots,C_{ij}\right) = C_{ij} + u_i\left(b_{j+k} - b_j\right)$$

Given $\mathscr{D}_t = \{Y_{ij} : i + j \le t\}$, under the assumptions BF1) and BF2), we have

$$\Rightarrow \qquad E\left(C_{it} \middle| \mathscr{D}_t\right) = C_{i,\ t-i} + u_i\left(1 - b_{t-i}\right) \qquad i = 0,\ldots,t$$

The **BF estimator for the ultimate claims** C_{it} is

$$\tilde{C}_{it}^{BF} = C_{i,\, t-i} + \tilde{u}_i \left(1 - \tilde{b}_{t-i}\right)$$

where

$\tilde{u}_i$, $i = 0,\ldots,t$, and $\tilde{b}_j$, $j = 0,\ldots,t$, denote some appropriate estimators of the parameters.

Since $E(C_{it}) = u_i$, the estimates of the parameters u_i, $i = 0,\ldots,t$, are called "initial" estimates of the ultimate claims; typically, these estimates are based on external data, e.g. from pricing or market information.

As for the estimate of the development pattern, the CL development factors $\hat{f}_j$ are generally used.

Remark:

➢ this approach can be motivated by assuming for the incremental payments a GLM with overdispersed Poisson distribution.

Overdispersed Poisson (ODP) GLM (Alai et al. (2009))

- the incremental payments Y_{ij}, $i, j = 0,\ldots,t$,are stochastically independent and overdispersed Poisson distributed with parameters $u_i > 0$, $i = 0,\ldots,t$, and $\gamma_j > 0$, $j = 0,\ldots,t$, such that

  $E(Y_{ij}) = u_i \gamma_j$, $\qquad var(Y_{ij}) = \phi\, E(Y_{ij})$

  for all $i, j = 0,\ldots,t$, and $\sum_{j=0}^{t} \gamma_j = 1$ (normalizing condition)
- $\tilde{u}_i$, $i = 0,\ldots,t$, are independent and unbiased estimators for $u_i = E(C_{it})$
- Y_{ij} and $\tilde{u}_k$ are independent for all i, j, k.

Remarks:

➢ Model assumption BF2) is fulfilled with $b_j = \sum_{k=0}^{j} \gamma_k$. Hence the ODP model can be used to explain the BF model;

➢ The $\tilde{u}_i$, $i = 0,\ldots,t$, are some "initial" estimators of the expected ultimate claims. It is assumed that such estimates are done prior to the observation of the run-off data. Henceforth, they are exogenous estimates based only on external data and expert opinion.

We have denoted

$$\tilde{\gamma}_j^{POI}, \qquad j = 0, \ldots, t$$

the maximum likelihood (ML) estimators, solution of the log-likelihood equations of the Poisson model.

If we use such (ML) estimators for the estimation of the development pattern γ_j we obtain the following **BF estimator**

$$\tilde{C}_{it}^{BF} = C_{i,\ t-i} + \tilde{u}_i \left(1 - \sum_{j=0}^{t-i} \tilde{\gamma}_j^{POI} \right)$$

from which the following estimate is obtained

$$\hat{C}_{it}^{BF} = C_{i,\ t-i} + \hat{u}_i \left(1 - \sum_{j=0}^{t-i} \hat{\gamma}_j^{POI} \right)$$

If we recall that

$$\sum_{j=0}^{t-i} \hat{\gamma}_j^{POI} = \hat{b}_{t-i}^{CL} = \prod_{k=t-i}^{t-1} \hat{f}_k^{-1}$$

where $\hat{f}_k,\ k = 0, \ldots, t-1$, are the CL estimates of the CL factors, we note that the estimate $\hat{C}_{it}^{BF}$ is just that usually used for the BF method in practice.

MSEP in the BF method, single origin year

Let the **BF estimator for the claims reserve** $R_i = C_{it} - C_{i,t-i}$

$$\tilde{R}_i^{BF} = \tilde{u}_i \left(1 - \sum_{j=0}^{t-i} \tilde{\gamma}_j^{POI} \right)$$

We have to evaluate the **conditional mean square error of prediction** of the BF estimators of the claims reserves $\tilde{R}_i^{BF} = \tilde{C}_{it}^{BF} - C_{i,t-i}$

$$MSEP_{\mathscr{D}_t}(\tilde{R}_i^{BF}) = E\left[(\tilde{R}_i^{BF} - R_i)^2 \middle| \mathscr{D}_t \right] = E\left[(\tilde{C}_{it}^{BF} - C_{i,t-i} - C_{it} + C_{i,t-i})^2 \middle| \mathscr{D}_t \right]$$

$$= E\left[(\tilde{C}_{it}^{BF} - C_{it})^2 \middle| \mathscr{D}_t \right] = MSEP_{\mathscr{D}_t}(\tilde{C}_{it}^{BF})$$

Following Alai *et al*. (2010) we get

$$MSEP_{\mathscr{D}_t}(\tilde{C}_{it}^{BF}) = \sum_{j=t-i+1}^{t} var(Y_{ij}) + \left(1 - \sum_{j=0}^{t-i} \tilde{\gamma}_j^{POI}\right)^2 var(\tilde{u}_i) + u_i^2 \left(\sum_{j=t-i+1}^{t} \gamma_j - \sum_{j=t-i+1}^{t} \tilde{\gamma}_j^{POI}\right)^2$$

They suggest to estimate the three terms as follows.

The process variance $\sum_{j=t-i+1}^{t} var(Y_{ij})$ is estimated by using the following estimates

$$v\hat{a}r(Y_{ij}) = \hat{\phi}\,\hat{u}_i \hat{\gamma}_j^{POI}$$

For the second term, some estimates of the uncertainty in the "initial" estimates $\tilde{u}_i$ are needed and they can be obtained exogenously. For instance, the coefficient of variation of the estimator $\tilde{u}_i$ could be given. Hence, the following estimate could be considered

$$\left(1 - \sum_{j=0}^{t-i} \hat{\gamma}_j^{POI}\right)^2 \hat{u}_i^2 \left(\widehat{Vco}(\tilde{u}_i)\right)^2$$

where $\widehat{Vco}(\tilde{u}_i)$ is an estimate of the coefficient of variation of the estimator $\tilde{u}_i$.

Much more complicated is to estimate the following quantity in the third term

$$\left(\sum_{j=t-i+1}^{t} \gamma_j - \sum_{j=t-i+1}^{t} \tilde{\gamma}_j^{POI}\right)^2$$

The standard approach is to estimate this quantity by the unconditional expectation

$$E\left[\left(\sum_{j=t-i+1}^{t} (\gamma_j - \tilde{\gamma}_j^{POI})\right)^2\right] = \sum_{j,l=t-i+1}^{t} E\left[(\gamma_j - \tilde{\gamma}_j^{POI})(\gamma_l - \tilde{\gamma}_l^{POI})\right]$$

Recalling the asymptotic properties of the ML estimators $\tilde{\boldsymbol{\beta}}$

$$\tilde{\boldsymbol{\beta}} \approx N(\hat{\boldsymbol{\beta}}, [\mathscr{I}(\hat{\boldsymbol{\beta}})]^{-1})$$

where $\mathscr{I}(\boldsymbol{\beta}) = E\left[-\frac{\partial^2 \tilde{\ell}}{\partial\beta_j \partial\beta_h}\right]_{j,h}$ is the Fischer information matrix,

we have

$$\hat{E}(\tilde{\gamma}_j^{POI}) = \hat{\gamma}_j^{POI}$$

Therefore, by applying the following approximations,

$$E\left[(\gamma_j - \tilde{\gamma}_j^{POI})(\gamma_l - \tilde{\gamma}_l^{POI})\right] \cong E\left[(\hat{\gamma}_j^{POI} - \tilde{\gamma}_j^{POI})(\hat{\gamma}_l^{POI} - \tilde{\gamma}_l^{POI})\right]$$

$$= E\left[(\hat{E}(\tilde{\gamma}_j^{POI}) - \tilde{\gamma}_j^{POI})(\hat{E}(\tilde{\gamma}_l^{POI}) - \tilde{\gamma}_l^{POI})\right] \cong E\left[(E(\tilde{\gamma}_j^{POI}) - \tilde{\gamma}_j^{POI})(E(\tilde{\gamma}_l^{POI}) - \tilde{\gamma}_l^{POI})\right]$$

the second term in the MSEP can be estimated as follows

$$\sum_{j,l=t-i+1}^{t} E\left[(\gamma_j - \tilde{\gamma}_j^{POI})(\gamma_l - \tilde{\gamma}_l^{POI})\right] \approx \sum_{j,l=t-i+1}^{t} Cov(\tilde{\gamma}_j^{POI}, \tilde{\gamma}_l^{POI})$$

Recalling again the asymptotic properties of the ML estimators $\tilde{\boldsymbol{\beta}}$, for the covariance $Cov(\tilde{\gamma}_j^{POI}, \tilde{\gamma}_l^{POI})$ the estimate provided by the element of the estimated Fisher information matrix denoted by g_{jl} is used.

Hence, the following estimate for the MSEP is obtained

$$M\hat{S}EP_{\mathscr{D}_t}(\tilde{C}_{it}^{BF}) = \sum_{j=t-i+1}^{t} \hat{\phi}\,\hat{u}_i \hat{\gamma}_j^{POI} + \left(1 - \sum_{j=0}^{t-i} \hat{\gamma}_j^{POI}\right)^2 \hat{u}_i^2 \left(\widehat{Vco}(\tilde{u}_i)\right)^2 + \hat{u}_i^2 \sum_{j,l=t-i+1}^{t} g_{jl}$$

MSEP in the BF method, aggregated origin years

Denoted by R the **total claims reserve**

$$R = \sum_{i=1}^{t} R_i$$

the **BF estimator for the total claims reserve** is

$$\tilde{R}^{BF} = \sum_{i=1}^{t} \tilde{R}_i^{BF} = \sum_{i=1}^{t}\left(\tilde{C}_{it}^{BF} - C_{i,t-i}\right)$$

We have to evaluate the **conditional mean square error of prediction** of the BF estimator of the total claims reserve $\tilde{R}^{BF}$

$$MSEP_{\mathscr{D}_t}(\tilde{R}^{BF}) = E\left[\left(\sum_{i=1}^{t} \tilde{R}_i^{BF} - \sum_{i=1}^{t} R_i\right)^2 \middle| \mathscr{D}_t\right] = E\left[\left(\sum_{i=1}^{t} \tilde{C}_{it}^{BF} - \sum_{i=1}^{t} C_{it}\right)^2 \middle| \mathscr{D}_t\right] = MSEP_{\mathscr{D}_t}\left(\sum_{i=1}^{t} \tilde{C}_{it}^{BF}\right)$$

Consider two different origin years $i < k$

$$MSEP_{\mathscr{D}_t}\left(\tilde{C}_{it}^{BF} + \tilde{C}_{kt}^{BF}\right) = E\left[(\tilde{C}_{it}^{BF} + \tilde{C}_{kt}^{BF} - C_{it} - C_{kt})^2 \middle| \mathscr{D}_t\right]$$

$$= var(C_{it} + C_{kt} | \mathscr{D}_t) + \left(\tilde{C}_{it}^{BF} + \tilde{C}_{kt}^{BF} - E(C_{it} + C_{kt} | \mathscr{D}_t)\right)^2$$

Because of the independence assumption

$$var(C_{it} + C_{kt} | \mathscr{D}_t) = var(C_{it} | \mathscr{D}_t) + var(C_{kt} | \mathscr{D}_t)$$

For the conditional parameter estimation error we have

$$\left(\tilde{C}_{it}^{BF} + \tilde{C}_{kt}^{BF} - E(C_{it}|\mathscr{D}_t) - E(C_{kt}|\mathscr{D}_t)\right)^2 = \left(\tilde{C}_{it}^{BF} - E(C_{it}|\mathscr{D}_t)\right)^2 + \left(\tilde{C}_{kt}^{BF} - E(C_{kt}|\mathscr{D}_t)\right)^2 +$$
$$+ 2\left(\tilde{C}_{it}^{BF} - E(C_{it}|\mathscr{D}_t)\right)\left(\tilde{C}_{kt}^{BF} - E(C_{kt}|\mathscr{D}_t)\right)$$

Hence

$$MSEP_{\mathscr{D}_t}\left(\tilde{C}_{it}^{BF} + \tilde{C}_{kt}^{BF}\right) = MSEP_{\mathscr{D}_t}\left(\tilde{C}_{it}^{BF}\right) + MSEP_{\mathscr{D}_t}\left(\tilde{C}_{kt}^{BF}\right) +$$
$$+ 2\left(\tilde{C}_{it}^{BF} - E(C_{it}|\mathscr{D}_t)\right)\left(\tilde{C}_{kt}^{BF} - E(C_{kt}|\mathscr{D}_t)\right)$$

So, in the $MSEP_{\mathscr{D}_t}(\tilde{R}^{BF})$ we have to estimate all the cross-products.

The $MSEP_{\mathscr{D}_t}(\tilde{R}^{BF})$ is defined as follows (Alai et al. (2010)

$$MSEP_{\mathscr{D}_t}(\tilde{R}^{BF}) = MSEP_{\mathscr{D}_t}\left(\sum_{i=1}^{t} \tilde{C}_{it}^{BF}\right)$$

$$= \sum_{i=1}^{t} MSEP_{\mathscr{D}_t}\left(\tilde{C}_{it}^{BF}\right) + 2\sum_{i<k} u_i u_k \sum_{\substack{j>t-i \\ l>t-k}} E\left[(\gamma_j - \tilde{\gamma}_j^{POI})(\gamma_l - \tilde{\gamma}_l^{POI})\right]$$

As above, for the estimation of the cross-products the estimates of the covariances $Cov(\tilde{\gamma}_j^{POI}, \tilde{\gamma}_l^{POI})$ are used.

So we get the following estimate:

$$M\hat{S}EP_{\mathscr{D}_t}(\tilde{R}^{BF}) = \sum_{i=1}^{t} M\hat{S}EP_{\mathscr{D}_t}(\tilde{R}_i^{BF}) + 2\sum_{i<k} \hat{u}_i \hat{u}_k \sum_{\substack{j>t-i \\ l>t-k}} g_{jl}$$

where g_{jl} denotes the element of the estimated Fisher information matrix that provides and estimate for the covariance $Cov(\tilde{\gamma}_j^{POI}, \tilde{\gamma}_l^{POI})$.

References

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THE BORNHUETTER-FERGUSON MODEL AND GLM

BF model (Wüthrich, Merz (2008))

BF1) the random vectors $(C_{i0},\dots,C_{it})$, $i=0,\dots,t$ are stochastically independent

BF2) There exist parameters $u_i > 0$, $i=0,\dots,t$, $b_j > 0$, $j=0,\dots,t$ with $b_t = 1$, such that for all $i=0,\dots,t$, $j=0,\dots,t-1$ and $k=0,\dots,t-j$, we have

$$E(C_{i0}) = u_i b_0$$

$$E\left(C_{i,\,j+k} \middle| C_{i0},\dots,C_{ij}\right) = C_{ij} + u_i\left(b_{j+k} - b_j\right)$$

The **BF estimator for the ultimate claims** C_{it} is $\tilde{C}_{it}^{BF} = C_{i,\,t-i} + \tilde{u}_i\left(1-\tilde{b}_{t-i}\right)$

where, in the practice,

$\tilde{u}_i$, $i=0,\dots,t$, provide the "initial" estimates of the ultimate claims

$\tilde{b}_j$, $j=0,\dots,t$, are the estimators provided by the CL development factors.

Mack (2006) criticizes the practice of estimating the claims development pattern by $\hat{b}_j^{CL}$, $j = 0, \ldots, t-1$, because in doing so the hypothesis of the CL method are assumed implicitly and it is contradicted the basic idea of independence between the last observed cumulative claims $C_{i,\, t-i}$ and estimated outstanding claims liabilities $\tilde{C}_{it}^{BF} - C_{i,\, t-i}$, which was fundamental to the origin of the BF method.

Mack proposes the following estimates of the parameters b_j, $j = 0, \ldots, t$:

$$\hat{b}_j^M = \hat{\gamma}_0^M + \ldots + \hat{\gamma}_j^M ,$$

with $$\hat{\gamma}_j^M = \frac{\sum_{i=0}^{t-j} y_{ij}}{\sum_{i=0}^{t-j} \hat{u}_i}, \qquad j = 0, \ldots, t.$$

Since such estimates do not ensure that $\hat{b}_t^M = \hat{\gamma}_0^M + \ldots + \hat{\gamma}_t^M = 1$, Mack (2006, 2008) suggests to consider them as raw estimates, which have to be submitted to some smoothing and extrapolating procedure in order to get an estimate $\hat{\gamma}_{t+1}$ of the tail ratio and to fulfil the constraint $\hat{\gamma}_0 + \ldots + \hat{\gamma}_{t+1} = 1$.

A GLM for the BF model

- Response variables: incremental payments Y_{ij}, $i, j = 0, \ldots, t$, stochastically independent and overdispersed Poisson distributed

- Covariates or explanatory variables: origin year, development year

$$\eta_{ij} = \mathbf{x}'_{ij} \boldsymbol{\beta} = \alpha_i + \beta_j$$

- Link function: $g = \log$.

We get a multiplicative model for the expectation

$$E(Y_{ij}) = \exp(\mathbf{x}'_{ij} \boldsymbol{\beta}) = \exp(\alpha_i) \exp(\beta_j) = a_i \gamma_j$$

where $a_i = \exp(\alpha_i)$ and $\gamma_j = \exp(\beta_j)$.

Let $B_j = \sum_{k=0}^{j} \gamma_k$, $j = 0, \ldots, t$, then we have

$$E(C_{it}) = E\left(\sum_{j=0}^{t} Y_{ij} \right) = a_i \sum_{j=0}^{t} \gamma_j = a_i B_t$$

$$E(R_i) = E\left(\sum_{j=t-i+1}^{t} Y_{ij} \right) = a_i \sum_{j=t-i+1}^{t} \gamma_j = a_i (B_t - B_{t-i}) = a_i B_t \left(1 - \frac{B_{t-i}}{B_t} \right).$$

Let $\hat{\alpha}_i$, $i = 0,\ldots,t$, and $\hat{\beta}_j$, $j = 0,\ldots,t$, the quasi-maximum likelihood estimates, it can be shown that the prevision of the claims reserve coincides with the CL claims reserve

$$\hat{R}_i^{ML} = \hat{a}_i \hat{B}_t \left(1 - \frac{\hat{B}_{t-i}}{\hat{B}_t}\right) = \hat{R}_i^{CL}$$

where $\hat{a}_i = \exp(\hat{\alpha}_i)$, $\hat{B}_j = \sum_{k=0}^{j} \hat{\gamma}_k$ and $\hat{\gamma}_j = \exp(\hat{\beta}_j)$.

Let $\tilde{R}_i^{ML}$ the corresponding estimator

$$\tilde{R}_i^{ML} = \sum_{j=t-i+1}^{t} \exp(\tilde{\alpha}_i + \tilde{\beta}_j) = \tilde{a}_i \tilde{B}_t \left(1 - \frac{\tilde{B}_{t-i}}{\tilde{B}_t}\right)$$

Remark:

In Alai et al. (2009), the reserve estimate, coherent with traditional BF approach, is

$$\hat{R}_i^{BF} = \hat{u}_i \left(1 - \frac{\hat{B}_{t-i}}{\hat{B}_t}\right)$$

where $\hat{u}_i$ is the initial estimate of the ultimate claims of origin year i and $\hat{B}_{t-i} / \hat{B}_t$ are the ODP estimate of the claims development pattern.

Remarks:

- Alai et al. (2009) point out that this approach could appear in some sense inconsistent. In fact, after estimating the parameters $\hat{\alpha}_i$ and $\hat{\beta}_j$ by an ODP model, only the estimates $\hat{\beta}_j$, $j = 0,\ldots,t$, are used. However, this is what practitioners do when they apply the BF method.
- Recall that, according to Mack (2006), the use of the CL development factors contradicts the basic idea fundamental to the origin of the BF method (i.e. the independence between the last observed cumulative claims and estimated reserve).

Now, we will introduce a quasi-likelihood model, in which only the parameters β_j, $j = 0,\ldots,t$, related to the development years, will be estimated and the parameter estimates will take account of the external estimates $\hat{u}_i$, $i = 0,\ldots,t$, too.

Since in the quasi-likelihood ODP model for the incremental payments we have

$$E(C_{ij}) = E\left(\sum_{k=0}^{j} Y_{ik}\right) = \exp(\alpha_i)\sum_{k=0}^{j}\exp(\beta_k) = a_i \sum_{k=0}^{j}\gamma_k = a_i B_j$$

whereas in the BF method, we have

$$E(C_{ij}) = u_i b_j$$

to take account, in the GLM, of the external estimates $\hat{u}_i$, $i = 0,\ldots,t$, we can set the parameters related to the accident years $a_i = \hat{u}_i$.

Note that, whereas in the BF it is $b_t = 1$ and $E(C_{it}) = u_i$, in the GLM it is not assured that $B_t = 1$, unless it is set as a constraint.

So in the GLM we have

$$E(C_{it}) = a_i B_t$$

where B_t can be interpreted as an adjusting factor taking account of the run-off data.

A GLM with offset terms for the external estimates (Gigante et al. (2010))

- Response variables: incremental payments Y_{ij}, $i, j = 0,\ldots,t$, stochastically independent and overdispersed Poisson distributed
- Covariates or explanatory variables: development years; origin years set as offset terms (i.e. covariates with known effects), $a_i = \hat{u}_i$

$$\eta_{ij} = \log \hat{u}_i + \beta_j$$

- Link function: $g = \log$.

We get

$$E(Y_{ij}) = \exp(\mathbf{x}'_{ij}\boldsymbol{\beta}) = \exp(\alpha_i)\exp(\beta_j) = \hat{u}_i \gamma_j$$

where $\alpha_i = \log \hat{u}_i$ and $\gamma_j = \exp(\beta_j)$.

The quasi-likelihood estimates of the regression parameters β_j, $j = 0,\ldots,t$, are the solutions $\hat{\beta}_j^{OFS}$ of the following equations:

$$\sum_{i=0}^{t-j} y_{ij} = \sum_{i=0}^{t-j} \hat{u}_i \exp(\beta_j), \qquad j = 0,\ldots,t.$$

Denoting by

$$\hat{\gamma}_j^{OFS} = \exp\left(\hat{\beta}_j^{OFS}\right) = \frac{\sum_{i=0}^{t-j} y_{ij}}{\sum_{i=0}^{t-j} \hat{u}_i}, \qquad j = 0, \ldots, t,$$

we get the following estimates of the parameters B_j: $\quad \hat{B}_j^{OFS} = \sum_{k=0}^{j} \hat{\gamma}_k^{OFS}, \qquad j = 0, \ldots, t$

which coincide with the estimates suggested by Mack (2006).

The estimator of the claims reserve of origin year i is

$$\tilde{R}_i^{OFS} = \hat{u}_i \sum_{j=t-i+1}^{t} \exp\left(\tilde{\beta}_j\right) = \hat{u}_i \tilde{B}_t^{OFS} \left(1 - \frac{\tilde{B}_{t-i}^{OFS}}{\tilde{B}_t^{OFS}}\right)$$

and the estimate of the claims reserve is

$$\hat{R}_i^{OFS} = \hat{u}_i \sum_{j=t-i+1}^{t} \hat{\gamma}_j^{OFS} = \hat{u}_i \hat{B}_t^{OFS} \left(1 - \frac{\hat{B}_{t-i}^{OFS}}{\hat{B}_t^{OFS}}\right).$$

Remark:
Notice that $\hat{u}_i \hat{B}_t^{OFS}$ can be interpreted as an estimate of the ultimate claims of origin year i, "updated" according to the data, whereas $\hat{B}_{t-i}^{OFS} / \hat{B}_t^{OFS}$ is the estimate of the rate of the ultimate claims paid within development year $t-i$.

The Bornhuetter-Ferguson model and GLM

MSEP, single origin year

We have to evaluate the

$$MSEP_{\hat{u}}\left(\tilde{R}_i^{OFS}\right) = var\left(R_i\right) + E[(\tilde{R}_i^{OFS} - E(R_i))^2]$$

where

$$R_i = \sum_{j=t-i+1}^{t} Y_{ij} \qquad \text{and} \qquad \tilde{R}_i^{OFS} = \hat{u}_i \sum_{j=t-i+1}^{t} \exp\left(\tilde{\beta}_j\right)$$

Remark:
The MSEP depends on the vector $\hat{\boldsymbol{u}} = (\hat{u}_0, \hat{u}_1, \ldots, \hat{u}_t)'$ of the external estimates of the ultimate claims.

An estimate of the first term is

$$\hat{var}\left(R_i\right) = \sum_{j=t-i+1}^{t} \hat{var}(Y_{ij}) = \sum_{j=t-i+1}^{t} \hat{\phi}\, \hat{u}_i\, \hat{\gamma}_j^{OFS}.$$

The second term can be estimated by means of the delta method.

If $\tilde{\beta}$ is a maximum likelihood estimator,
h is a regular function and
the number of observations is sufficiently high,

then

the distribution of $h(\tilde{\beta})$ can be approximated by the normal distribution

$$h(\tilde{\beta}) \approx N(h(\hat{\beta}), J_h(\hat{\beta})[\mathscr{I}(\hat{\beta})]^{-1} J_h(\hat{\beta})')$$

where $J_h(\hat{\beta})$ is the jaconian matrix of h and
$[\mathscr{I}(\hat{\beta})]^{-1}$ is the inverse of the Fisher information matrix evaluated at the maximum likelihood estimate $\hat{\beta}$.

Since the estimator $\tilde{R}_i^{OFS}$ is a function of the maximum likelihood estimator $\tilde{\boldsymbol{\beta}}_{DY}$ of the regression parameters $\boldsymbol{\beta}_{DY} = (\beta_0, ..., \beta_t)$ relating to the development years, we can write

$$\tilde{R}_i^{OFS} = f\left(Y_{ij} : i + j \le t\right) = h_i(\tilde{\boldsymbol{\beta}}_{DY})$$

where h_i is the function $h_i(\boldsymbol{\beta}_{DY}) = \hat{u}_i \sum_{j=t-i+1}^{t} \exp\left(\beta_j\right).$

The Bornhuetter-Ferguson model and GLM

By applying the delta method we have

$$\hat{E}(h_i(\tilde{\boldsymbol{\beta}}_{DY})) = h_i(\hat{\boldsymbol{\beta}}_{DY}),$$

and from GLM assumptions we can set

$$\hat{E}(R_i) = h_i(\hat{\boldsymbol{\beta}}_{DY})$$

Therefore, by applying the following approximations,

$$E[(\tilde{R}_i^{OFS} - E(R_i))^2] \cong E[(\tilde{R}_i^{OFS} - \hat{E}(R_i))^2] = E[(\tilde{R}_i^{OFS} - h_i(\hat{\boldsymbol{\beta}}_{DY}))^2] \cong E[(\tilde{R}_i^{OFS} - E(\tilde{R}_i^{OFS}))^2]$$

the second term in the MSEP can be estimated by the variance of the estimator $\tilde{R}_i^{OFS}$.

Hence, again by the delta method, we have

$$MS\hat{E}P_{\hat{u}}\left(\tilde{R}_i^{OFS}\right) = v\hat{a}r(R_i) + J_h(\hat{\beta})[\mathscr{I}(\hat{\beta})]^{-1} J_h(\hat{\beta})'.$$

MSEP in the BF method, aggregated origin years

To evaluate the **mean square error of prediction**

$$MSEP_{\hat{u}}\left(\tilde{R}^{OFS}\right) = var(R) + E[(\tilde{R}^{OFS} - E(R))^2]$$

where

$$R = \sum_{i=1}^{t} \sum_{j=t-i+1}^{t} Y_{ij} \quad \text{and} \quad \tilde{R}^{OFS} = \sum_{i=1}^{t} \hat{u}_i \sum_{j=t-i+1}^{t} \exp\left(\tilde{\beta}_j\right)$$

we estimate the first term by

$$v\hat{a}r(R) = \sum_{i=1}^{t} \sum_{j=t-i+1}^{t} v\hat{a}r(Y_{ij}) = \sum_{i=1}^{t} \sum_{j=t-i+1}^{t} \hat{\phi}\hat{u}_i \, \hat{\gamma}_j^{OFS} .$$

To estimate the second term we apply again the delta method, by defining the function

$$h(\boldsymbol{\beta}_{DY}) = \sum_{i=1}^{t} \hat{u}_i \sum_{j=t-i+1}^{t} \exp\left(\beta_j\right)$$

Example

Data: Wüthrich, Merz, (2008).

Accident year i	$\hat{R}_i^{OFS}$	RMSEP	RMSEP %	Process risk %	Estimation risk %
1	15,425	26,237	170.1%	121.0%	119.5%
2	25,723	31,843	123.8%	93.7%	80.9%
3	35,833	35,952	100.3%	79.4%	61.3%
4	90,629	53,560	59.1%	49.9%	31.6%
5	168,014	70,901	42.2%	36.7%	20.9%
6	319,288	94,947	29.7%	26.6%	13.3%
7	531,806	120,134	22.6%	20.6%	9.2%
8	1,199,795	177,002	14.8%	13.7%	5.4%
9	4,257,538	329,699	7.7%	7.3%	2.6%
Total	6,644,050	489,453	7.4%	5.8%	4.5%

Table 4: process risk and estimation risk of the claims reserve estimators $\tilde{R}_i^{OFS}$

Remarks:

- We have estimated $\hat{B}_t^{OFS} = 0.88523 < 1$, then $\hat{u}_i \hat{B}_t^{OFS} < \hat{u}_i$, i.e. the "updated" estimate of the ultimate claims of accident year i is lower than the initial estimate $\hat{u}_i$, for all i.
- RMSEP% is the evaluations of the prediction errors (RMSEP), given by the square root of the MSEP, expressed as a percentage of the claims reserve estimate.
- As for the process risk and the estimation risk, for ease of comparison, the corresponding percentages of the square root of the two terms in the MSEP are reported.
- As typically happens there is a considerable uncertainty in the reserve estimates, in particular in the early accident years, where the claims reserve estimates are small. Then, the prediction error, as a percentage of the reserve estimate decreases.
- The very high errors depend also on the Pearson estimate of the dispersion parameter $\hat{\phi} = 22{,}591$.
- The mean squared error of prediction depends on the vector $\hat{\boldsymbol{u}} = (\hat{u}_0, \hat{u}_1, \ldots, \hat{u}_t)'$, but it does not take account of the intrinsic variability of such estimates.

In order to evaluate this aspect too, we follow a simulation approach (25,000 simulations) by assuming for $\tilde{u}_i$ the gamma distribution with expectation $\hat{u}_i$. For the sake of comparison with the example in Alai et al. (2009), we assume a coefficient of variation for $\tilde{u}_i$ of 0.05.

Claims Reserving methods	Claims reserve estimates	RMSEP (%)	Process Risk (%)	Estimation risk (%)
$\hat{R}^{OFS}$ - $\hat{\phi} = 22{,}591$	6,644,573	539,469 (8.1%)	426,278 (6.4%)	330,627 (5.0%)
CL method	6,047,061	462,960 (7.7%)	424,379 (7.0%)	185,026 (3.1%)
BF Alai *et al.* (2009) - $\hat{\phi} = 14{,}714$	7,356,575	471,971 (6.4%)	329,007 (4.5%)	338,396 (4.6%)
BF Mack (2008)	7,505,506	726,431 (9.7%)	621,899 (8.3%)	375,424 (5.0%)

Remarks:

$$\hat{R}_i^{OFS} = \hat{u}_i \hat{B}_t^{OFS}\left(1-\frac{\hat{B}_{t-i}^{OFS}}{\hat{B}_t^{OFS}}\right) \qquad \hat{R}_i^{CL} = \hat{R}_i^{ML} = \hat{a}_i \hat{B}_t\left(1-\frac{\hat{B}_{t-i}}{\hat{B}_t}\right)$$

$$\hat{R}_i^{BF} = \hat{u}_i\left(1-\frac{\hat{B}_{t-i}}{\hat{B}_t}\right)$$

$$\hat{R}_i^{M} = \hat{u}_i\left(1-\frac{\hat{B}_{t-i}^{OFS}}{\hat{B}_t^{OFS}}\right),$$

where $\hat{B}_j^{OFS}$, $j = 0,\ldots,t$, coincide with the estimates of the parameters b_j of the BF method, suggested by Mack(2006); recall that we have $\hat{B}_t^{OFS} < 1$.

References

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A STOCHASTIC CLAIMS RESERVING MODEL WITH RANDOM EFFECTS

Let

- Y_{ij}, $i, j = 0,\ldots,t$, ***response variables*** (incremental payments)
- $\boldsymbol{U} = (U_0,\ldots,U_t)$ ***unobservable risk parameters*** related to the origin years,

1) ***Independence assumptions***

$(U_0, Y_{00},\ldots,Y_{0t}),\ldots,(U_t, Y_{t0},\ldots,Y_{tt})$ are independent,

$Y_{ij} | \boldsymbol{U} = \boldsymbol{u}$, $i, j = 0,\ldots,t$, are independent for any value $\boldsymbol{u}$ of $\boldsymbol{U}$,

$(Y_{i0},\ldots,Y_{it}) | \boldsymbol{U} = \boldsymbol{u} \overset{d}{=} (Y_{i0},\ldots,Y_{it}) | U_i = u_i$ for any $\boldsymbol{u} = (u_0,\ldots,u_t)$

2) ***Distributional assumptions for the responses conditional on risk parameters***

$Y_{ij} | U_i = u_i \sim$ ODP, $\quad E(Y_{ij} | U_i = u_i) = \mu_{ij} = u_i \exp(\beta_j)$, $\quad Var(Y_{ij} | U_i = u_i) = \phi\mu_{ij}$

3) ***Distributional assumptions for the risk parameters***

$U_i \sim$ Gamma, $\qquad E(U_i) = \psi_i$, $\qquad \mathrm{var}(U_i)/E(U_i) = \lambda_i$

where ψ_i and λ_i ***hyperparameters***; λ_i ***dispersion parameter***

Remarks: we have defined an **ODP-gamma model**

- $E(Y_{ij} | U_i = u_i) = u_i \exp(\beta_j) = \exp(\beta_j + w_i)$, with $w_i = \log(u_i)$
- $W_i = \log(U_i)$ follows a distribution conjugate of the ODP of $Y_{ij} | U_i = u_i$,
- The assumptions 1), 2), 3) define a **HGLM** with β_j, **fixed effects**, w_i, **random effects**

It is well-known that (e.g. Bühlmann, Gisler (2005); Verrall (2004); Wüthrich (2007))

given the parameters β_j, $j = 0,\ldots,t$, and ϕ

the hyperparameters ψ_i and λ_i, $i = 0,\ldots,t$

$\mathcal{D}_t = \{ Y_{ij} : \quad i + j \le t \}$

$$\Rightarrow \quad E(U_i | \mathcal{D}_t) = z_i \frac{\sum_{j=0}^{t-i} Y_{ij}}{\sum_{j=0}^{t-i} \exp(\beta_j)} + (1 - z_i)\,\psi_i \qquad \text{where} \qquad z_i = \frac{\sum_{j=0}^{t-i} \exp(\beta_j)}{\sum_{j=0}^{t-i} \exp(\beta_j) + \frac{\phi}{\lambda_i}};$$

$$\Rightarrow \quad E(R_i | \mathcal{D}_t) = \left[z_i \frac{\sum_{j=0}^{t-i} Y_{ij}}{\sum_{j=0}^{t-i} \exp(\beta_j)} + (1 - z_i)\,\psi_i \right] \sum_{j=t-i+1}^{t} \exp(\beta_j)$$

Bayesian estimator of R_i

A MIXTURE OF CL AND BF CLAIMS RESERVES

Let $C_{ij} = \sum_{h=0}^{j} Y_{ih}$ the cumulative payments. Since $E(Y_{ij}) = E(U_i)\exp(\beta_j) = \psi_i \exp(\beta_j)$

- $E(C_{ij}) = \psi_i \sum_{h=0}^{j} \exp(\beta_h)$ and $E(C_{it}) = \psi_i \sum_{h=0}^{t} \exp(\beta_h)$ expected ultimate claims

We have (e.g. England, Verrall (2002); Wüthrich (2007)):

$$E(R_i | \mathcal{D}_t) = \left[z_i \frac{C_{i,t-i}}{b_{t-i}} + (1 - z_i)\, E(C_{it}) \right] (1 - b_{t-i})$$

where $b_j = \frac{\sum_{h=0}^{j} \exp(\beta_h)}{\sum_{h=0}^{t} \exp(\beta_h)}$, $j = 0,\ldots,t$ is the claims development pattern,

in fact $E(C_{ij}) = E(C_{it}) b_j$

We recall that (e.g. England, Verrall (2002); Verrall (2007); Wüthrich (2007)):

$$E(R_i|\mathcal{D}_t) = \left[z_i \frac{C_{i,t-i}}{b_{t-i}} + (1-z_i)\, E(C_{it}) \right](1-b_{t-i})$$

can be interpreted as a mixture of the CL and the BF estimators.

In fact, given the parameters

	Bayesian estimators
CL model (Mack(1993))	$\tilde{R}_i^{BCL} = \frac{C_{i,t-i}}{b_{t-i}}(1-b_{t-i})$
BF model (Mack (2008); Wüthrich, Merz (2008))	$\tilde{R}_i^{BBF} = \mu_i(1-b_{t-i})$

If, in the ODP-Gamma, CL and BF models, the parameters are such that the claims development pattern $b_j,\ j=0,\ldots,t$ is the same and $E(C_{it}) = \mu_i$

$$\Rightarrow \qquad E(R_i|\mathcal{D}_t) = z_i\, \tilde{R}_{it}^{BCL} + (1-z_i)\, \tilde{R}_{it}^{BBF}$$

Parameter estimates: plug-in estimates (e.g. CL link ratios); a h-likelihood approach.

CONJUGATE HIERARCHICAL GENERALIZED LINEAR MODELS
(Lee, Nelder, Pawitan (2006))

Parameter estimation

Lee, Nelder (1996) introduced for HGLM the ***hierarchical*** or ***h-likelihood***

$$h \equiv log\ f_{\boldsymbol{Y},\boldsymbol{W}} = l_{\boldsymbol{Y}|\boldsymbol{W}=\boldsymbol{w}} + l_{\boldsymbol{W}}$$

$$h(\boldsymbol{\beta},\boldsymbol{w};\phi,\boldsymbol{\lambda};\boldsymbol{y},\boldsymbol{\psi},\boldsymbol{\omega}) = \sum_{\substack{i,j \\ i+j\le t}} \frac{\omega_{ij}}{\phi}\left[y_{ij}(\beta_j+w_i) - b(\beta_j+w_i)\right] + \sum_{i=0}^{t} \frac{\omega_i}{\phi}\left[\psi_i w_i - b(w_i)\right]; \quad \omega_i = \phi/\lambda_i$$

log-likelihood of an **augmented GLM** for the run-off data $\boldsymbol{y}$ and the pseudo-data $\boldsymbol{\psi}$

If ϕ and λ are known, the maximum h-likelihood estimates of the fixed and random effects are given by the GLM estimates of the regression parameters $\boldsymbol{\delta} = (\boldsymbol{\beta}^T, \boldsymbol{w}^T)^T$ obtained by the usual Iterative Weighted Least Square (IWLS) algorithm.

The IWLS algorithm can be extended to the non-conjugate case (HGLM).

QUASI - HIERARCHICAL GENERALIZED LINEAR MODELS

- No full distributional assumptions are needed
- Regression structure for the dispersion parameters: $\phi_{ij} = g_{\phi}^{-1}(\boldsymbol{x}_{\phi,ij}^T \boldsymbol{\gamma}_{\phi}),\ \lambda_i = g_{\lambda}^{-1}(\boldsymbol{x}_{\lambda,i}^T \boldsymbol{\gamma}_{\lambda})$

The ODP-gamma model can be interpreted as a quasi-HGLM.

Parameter estimation

A quasi-HGLM can be fitted by **estimating iteratively three interconnected GLMs** or **quasi-GLMs**.

The first one is an augmented GLM for the run-off data $\boldsymbol{y}$ and the pseudo-data $\boldsymbol{\psi}$

At convergence, the inverse of the Fisher information matrix $\mathcal{I}(\hat{\boldsymbol{\delta}})^{-1}$ of this augmented GLM provides an estimate of the **variance-covariance matrix**

$$var\begin{pmatrix} \tilde{\boldsymbol{\beta}} \\ \tilde{\boldsymbol{w}} - \boldsymbol{W} \end{pmatrix}$$

PREDICTION AND PREDICTION ERROR FOR CREDIBLE CLAIMS RESERVES

For the claims reserve $R = \sum_{i=1}^{t} R_i = \sum_{i,j:i+j>t} Y_{ij}$ we have

$$\overline{R} = E(R|\boldsymbol{U}) = \sum_{i,j:i+j>t} \exp(\beta_j + W_i)$$

and consider the estimator

$$\tilde{R} = \sum_{i,j:i+j>t} \exp(\tilde{\beta}_j + \tilde{w}_i)$$

where $\tilde{\boldsymbol{\beta}}$ and $\tilde{\boldsymbol{w}}$ are the quasi-HGLM estimators of $\boldsymbol{\beta}$ and $\boldsymbol{W}$.

*The **conditional mean square error of prediction** is*

$$MSEP_{R|\mathcal{D}_t}(\tilde{R}) = E\left[\left(R - \tilde{R}\right)^2 |\mathcal{D}_t\right]$$

$$= E\left[Var(R|U)|\mathcal{D}_t\right] + var(\overline{R}|\mathcal{D}_t) + E\left[\left(E(\overline{R}|\mathcal{D}_t) - \tilde{R}\right)^2 |\mathcal{D}_t\right]$$

Estimates of the three terms are given in Gigante *et al.* (2010), by following Booth, Hobert (1998) and Lee, Ha (2009):

- $\hat{E}\left[Var(R|U)|\mathcal{D}_t\right] = \sum_{i,j:i+j>t} \hat{\phi}\exp(\hat{\beta}_j + \hat{w}_i)$
- $\widehat{var}(\overline{R}|\mathcal{D}_t) = \widehat{var}(r(\boldsymbol{W})|\mathcal{D}_t) \approx J_r(\boldsymbol{w})\ \boldsymbol{H}_{22}^{-1}\ J_r(\boldsymbol{w})^T\Big|_{\hat{\boldsymbol{\delta}}}$
- $\hat{E}\left[\left(E(\overline{R}|\mathcal{D}_t) - \tilde{R}\right)^2|\mathcal{D}_t\right] \approx \hat{E}\left[\left(f(\boldsymbol{\beta}) - f(\tilde{\boldsymbol{\beta}})\right)^2|\mathcal{D}_t\right] \approx J_f(\boldsymbol{\beta})\,\boldsymbol{G}^{-1}\ J_f(\boldsymbol{\beta})^T\Big|_{\hat{\boldsymbol{\delta}}}$

with

– $J_r(\boldsymbol{w})$ and $J_f(\boldsymbol{\beta})$ the Jacobian matrices of the functions

$$r(\boldsymbol{w}) = \sum_{i,j:i+j>t} \exp(\beta_j + w_i) \qquad f(\boldsymbol{\beta}) = \sum_{i,j:i+j>t} g^{-1}(\boldsymbol{x}_{ij}^T\boldsymbol{\beta} + \tilde{w}_i(\boldsymbol{\beta}))$$

where $\tilde{w}_i(\boldsymbol{\beta})$ is the HGLM estimator of w_i for a given $\boldsymbol{\beta}$

– $\mathcal{I}(\boldsymbol{\delta}) = \begin{bmatrix} \boldsymbol{H}_{11} & \boldsymbol{H}_{12} \\ \boldsymbol{H}_{12}^T & \boldsymbol{H}_{22} \end{bmatrix}, \qquad \mathcal{I}(\boldsymbol{\delta})^{-1} = \begin{bmatrix} \boldsymbol{G}^{-1} & \boldsymbol{F} \\ \boldsymbol{F}^T & \boldsymbol{C} \end{bmatrix}$

HGLM ESTIMATES FOR CREDIBLE CLAIMS RESERVES

Given: the data of the run-off triangle $y_{ij}\,,\quad i,j = 0,\ldots,t,\quad i+j \le t$

the “external data” $\mu_0^{(i)},\quad i = 0,\ldots,t$

We set $\psi_i = E(U_i) = \mu_0^{(i)},\quad i = 0,\ldots,t$

We estimate

the fixed effect parameters $\beta_j,\ j = 0,\ldots,t$

the random effects $w_i = \log(u_i),\ i = 0,\ldots,t$

the dispersion parameters ϕ and $\lambda_i,\ i = 0,\ldots,t$

Given the estimates of the dispersion parameters, the quasi-HGLM estimates of the fixed and random effects satisfy the following conditions:

$$\begin{cases} \exp(\hat{\beta}_j) = \sum_{i=0}^{t-j} y_{ij} \Big/ \sum_{i=0}^{t-j} \hat{u}_i & j = 0,\ldots,t \\ \hat{u}_i = \hat{z}_i \dfrac{\sum_{j=0}^{t-i} y_{ij}}{\sum_{j=0}^{t-i} \exp(\hat{\beta}_j)} + (1-\hat{z}_i)\psi_i & i = 0,\ldots,t, \end{cases} \qquad with\ \hat{z}_i = \frac{\sum_{j=0}^{t-i} \exp(\hat{\beta}_j)}{\sum_{j=0}^{t-i} \exp(\hat{\beta}_j) + \dfrac{\hat{\phi}}{\hat{\lambda}_i}}$$

Compare with

$$E(U_i|\mathcal{D}_t) = z_i \frac{\sum_{j=0}^{t-i} Y_{ij}}{\sum_{j=0}^{t-i} \exp(\beta_j)} + (1-z_i)\,\psi_i \qquad z_i = \frac{\sum_{j=0}^{t-i} \exp(\beta_j)}{\sum_{j=0}^{t-i} \exp(\beta_j) + \dfrac{\phi}{\lambda_i}}$$

HGLM estimates of the claims reserves

$$\hat{R}_i = \hat{u}_i \sum_{j=t-i+1}^{t} \exp(\hat{\beta}_j) = \left[\hat{z}_i \frac{\sum_{j=0}^{t-i} y_{ij}}{\sum_{j=0}^{t-i} \exp(\hat{\beta}_j)} + (1-\hat{z}_i)\psi_i \right] \sum_{j=t-i+1}^{t} \exp(\hat{\beta}_j)$$

$$= \left[\hat{z}_i \frac{\sum_{j=0}^{t-i} y_{ij}}{\hat{b}_{t-i}} + (1-\hat{z}_i)\psi_i \sum_{j=0}^{t} \exp(\hat{\beta}_j) \right] (1-\hat{b}_{t-i}) = \hat{z}_i \hat{R}_i^{BCL} + (1-\hat{z}_i)\hat{R}_i^{BBF}$$

where

- $\hat{b}_j = \sum_{h=0}^{j} \exp(\hat{\beta}_h) \Big/ \sum_{h=0}^{t} \exp(\hat{\beta}_h), \qquad j = 0,\ldots,t$
- $\hat{R}_i^{BCL} = \dfrac{\sum_{j=0}^{t-i} y_{ij}}{\hat{b}_{t-i}} (1-\hat{b}_{t-i}), \qquad \hat{R}_i^{BBF} = \left(\psi_i \sum_{j=0}^{t} \exp(\hat{\beta}_j) \right) (1-\hat{b}_{t-i})$
- $\psi_i \sum_{j=0}^{t} \exp(\hat{\beta}_j)$ is an estimate of $E(C_{it}) = \psi_i \sum_{h=0}^{t} \exp(\beta_h)$

NUMERICAL EXAMPLE

Data: Tables 2.4-2.5 in Wüthrich, Merz (2008) – Borhuetter-Ferguson method.

Example: ODP-Gamma, quasi-HGLM model with $\phi_{ij} = \phi$ and $\lambda_i = \lambda$.

Origin year i	Initial estimate $\psi_i/1000$	Quasi-HGLM		
		$\hat{u}_i/1000$	$\sum_{j>t-i} \exp(\hat{\beta}_j)$ × 100	$\hat{R}_i$
1	11,367	11,906	0.1277	15,199
2	10,963	11,799	0.2214	26,125
3	10,617	10,952	0.3183	34,857
4	11,045	11,159	0.7762	86,623
5	11,481	11,459	1.3909	159,377
6	11,414	11,006	2.6763	294,565
7	11,127	10,219	4.6062	470,703
8	10,987	10,190	10.6646	1,086,682
9	11,618	11,194	36.2808	4,061,355

$$\hat{R}_i = \hat{z}_i \hat{R}_i^{BCL} + (1-\hat{z}_i)\hat{R}_i^{BBF}$$

Origin year i	$\hat{R}_i$	Quasi-HGLM				$\hat{R}_i^{CL}$	$\hat{R}_i^{BF}$	$1-\hat{b}_{t-i}^{CL}$
		$\hat{z}_i$	$\hat{R}_i^{BCL}$	$\hat{R}_i^{BBF}$	$1-\hat{b}_{t-i}$			
1	15,199	0.7391	15,442	14,511	0.00145	15,125	16,124	0.00142
2	26,125	0.7389	26,780	24,274	0.00251	26,257	26,998	0.00246
3	34,857	0.7387	35,234	33,791	0.00361	34,538	37,575	0.00354
4	86,623	0.7377	86,939	85,734	0.00880	85,301	95,434	0.00864
5	159,377	0.7363	159,268	159,682	0.01578	156,494	178,023	0.01551
6	294,565	0.7334	290,603	305,462	0.03036	286,121	341,306	0.02990
7	470,703	0.7289	455,156	512,508	0.05225	449,167	574,090	0.05160
8	1,086,682	0.7138	1,052,603	1,171,674	0.12097	1,043,243	1,318,646	0.12002
9	4,061,355	0.6254	3,969,176	4,215,257	0.41154	3,950,815	4,768,385	0.41042

Remark: $\hat{R}_i^{BBF} = \left[\mu_0^{(i)} \sum_{j=0}^{t} \exp(\hat{\beta}_j)\right](1-\hat{b}_{t-i})$, $\hat{R}_i^{BF} = \mu_0^{(i)}\,(1-\hat{b}_{t-i}^{CL})$, $\sum_{j=0}^{t} \exp(\hat{\beta}_j) = 0.88159$

$\hat{\phi} = 14,895$ $\hat{\lambda} = 47,936$ $\hat{\phi}/\hat{\lambda} = 0.31$

Origin year *i*	Reserve	RMSEP	RMSEP %
1	15,199	21,082	1.387
2	26,125	26,155	1.001
3	34,857	28,674	0.823
4	86,623	42,357	0.489
5	159,377	55,987	0.351
6	294,565	74,221	0.252
7	470,703	92,566	0.197
8	1,086,682	142,204	0.131
9	4,061,355	312,042	0.077
Total	6,235,486	419,505	0.067

Model	Reserve	RMSEP	RMSEP %
HGLM	6,235,486	419,505	0.067
CL-ODP	6,047,061	429,891	0.071
BF(Alai *et al*. (2009))	7,356,575	471,971	0.064

	HGLM	CL-ODP
$\hat{\phi}$	14,895	14,714

Reference

Gigante P., Picech L., Sigalotti L. (2013), Prediction error for credible claims reserves: an *h*-likelihood approach, European Actuarial Journal, 3(2), 453-470.